

Lorentz Group applicable to Finite Crystals

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Abstract

The two-by-two scattering matrix for one-dimensional scattering processes is a three-parameter $Sp(2)$ matrix or its unitary equivalent. For one-dimensional crystals, it would be repeated applications of this matrix. The problem is how to calculate N repeated multiplications of this matrix. It is shown that the original $Sp(2)$ matrix can be written as a similarity transformation of Wigner's little group matrix which can be diagonalized. It is then possible to calculate the repeated applications of the original $Sp(2)$ matrix.

Besides the three-dimensional rotation matrix, one of the most commonly used matrices in physics is the two-by-two $Sp(2)$ matrix. First of all, it is isomorphic to the Lorentz group applicable to two space dimensions and one time variable. It is the fundamental building block for linear canonical transformations for phase-space approach to both classical and quantum mechanics [1]. The same is true for classical and quantum optics.

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In addition, the one-dimensional scattering or beam-transfer matrix takes the form of a $Sp(2)$ matrix or its unitary equivalent. Indeed, this $Sp(2)$ matrix takes a simple mathematical form. It is real and unimodular, and therefore has three independent parameters. The question then arises as how to handle the calculation when this matrix is applied repeatedly for finite one-dimensional crystals, including periodic potentials [2, 3] and multilayer optics [4, 5].

In spite of its fundamental role in those many branches of physics, its mathematics is not yet thoroughly understood. First of all, there does not seem to exist an established procedure for diagonalizing this matrix. It is not difficult to get its eigenvalues, but the matrix cannot be diagonalized by a rotation. Is it then possible to construct a similarity transformation which brings it to a diagonal form? If it is possible, the most immediate application is the scattering of beams in the above-mentioned one-dimensional crystals which can mathematically be formulated through beam transfer matrices.

In this report, we show that the most general form of the $Sp(2)$ matrix can be written as a similarity transformation of Wigner's little group matrix [6, 7]. This Wigner matrix takes three different forms depending on the parameters of the original $Sp(2)$ matrix. However, these forms are either diagonal or can be diagonalized by a rotation.

In most of the physical applications, the one-cycle beam transfer matrix can be written as

$$R_1 B R_2 \tag{1}$$

with

$$R_i = \begin{pmatrix} \cos(\theta_i/2) & -\sin(\theta_i/2) \\ \sin(\theta_i/2) & \cos(\theta_i/2) \end{pmatrix},$$

$$B = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}. \tag{2}$$

This is known as the Bargmann decomposition of the $Sp(2)$ matrix. The original three-parameter matrix is written as a product of three one-parameter matrices.

We can write the expression of Eq.(1) as

$$R_1 B R_2 = (D R) B (R D^{-1}), \tag{3}$$

where

$$R = \sqrt{R_1 R_2} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{4}$$

with

$$\theta = \frac{\theta_1 + \theta_2}{2},$$

and

$$D = \sqrt{R_1 R_2^{-1}} = \begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{pmatrix}, \quad (5)$$

with

$$\delta = \frac{\theta_1 - \theta_2}{2},$$

and

$$R_1 R_2 = R_2 R_1 = R^2, \quad R_1 = DR, \quad R_2 = RD^{-1}. \quad (6)$$

We can thus write the starting matrix of Eq.(1) as

$$R_1 B R_2 = D(RBR)D^{-1}. \quad (7)$$

This is a similarity transformation of (RBR) with respect to D .

In this report, we are interested in diagonalizing the expression of Eq.(7), and thus calculating $(R_1 B R_2)^N$. For this purpose, we point out that (RBR) can be written as a similarity transformation [8, 4]

$$RBR = SW S^{-1}, \quad (8)$$

where S is in the form of

$$S = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad (9)$$

and W is Wigner's little-group matrix which takes one of the following three forms [4].

$$\begin{aligned} R(\phi) &= \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \\ X(\chi) &= \begin{pmatrix} \cosh(\chi/2) & \sinh(\chi/2) \\ \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}, \\ E(\gamma) &= \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad E(\gamma) = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}. \end{aligned} \quad (10)$$

We can then write

$$(R_1 B R_2) = D(SWS^{-1})D^{-1} = (DS)W(DS)^{-1}. \quad (11)$$

Indeed, the most general form of the $Sp(2)$ matrix can be written as a similarity transformation of the Wigner's little-group matrix.

The immediate application of this formula is to calculate $(R_1 B R_2)^N$. Although two-by-two matrices are simple, it is not trivial to calculate this expression for a large number of N . On the other hand, from Eq.(11), it is straightforward to write

$$(R_1 B R_2)^N = (DS)W^N(DS)^{-1}, \quad (12)$$

while it is a simple matter to calculate W^N from the expression given in Eq.(10). The expression for W^N becomes $R(N\phi)$, $X(N\chi)$ or $E(N\gamma)$.

The remaining problem is to relate the parameters η of the S matrix and ϕ, χ or γ of the W matrix from λ of B in Eq.(2) and θ of R in Eq.(4). The calculation is straightforward [4, 9]. When $(\cosh \lambda \sin \theta - \sinh \lambda)$ is positive, the relation becomes $R(\theta)BR(\theta) = SR(\phi)S^{-1}$, and the result is

$$\begin{aligned} \cos(\phi/2) &= \cosh \lambda \cos \theta, \\ e^{2\eta} &= \frac{\cosh \lambda \sin \theta + \sinh \lambda}{\cosh \lambda \sin \theta - \sinh \lambda}. \end{aligned} \quad (13)$$

If $(\cosh \lambda \sin \theta - \sinh \lambda)$ is negative, we should use $X(\chi)$ as the little-group matrix, and write $R(\theta)BR(\theta) = SX(\chi)S^{-1}$. The result is

$$\begin{aligned} \cosh(\chi/2) &= \cosh \lambda \cos \theta, \\ e^{2\eta} &= \frac{\cosh \lambda \sin \theta + \sinh \lambda}{\sinh \lambda - \cosh \lambda \sin \theta}. \end{aligned} \quad (14)$$

When $(\cosh \lambda \sin \theta - \sinh \lambda)$ goes through zero while it makes a transition from a small negative to positive number, is negative, we should use $R(\theta)BR(\theta) = SE(\gamma)S^{-1}$, the result is

$$e^\eta \sin(\phi/2) = \gamma. \quad (15)$$

In this case, e^η becomes very large, and ϕ has to be very small for γ to be finite [4, 9].

In this report, we started with the most general form of the $Sp(2)$ matrix and its Bargmann decomposition. We then showed that it can also be written as a similarity transformation of Wigner's little group matrix, which

can be diagonalised by a rotation. The immediate application of this similarity transformation is in the one-dimensional crystals requiring repeated applications of the $Sp(2)$ matrix.

Although the $Sp(2)$ matrix is mathematically convenient, its physical application is often made through its unitary equivalent [10, 11]. Transformations among those equivalent representations are also mathematically challenging problems.

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